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THE POSSIBLE GEOMETRIC ORGINS OF THE PRINCIPLE OF HALF-SUMS AND HALF-DIFFERENCES OF UNKNOWN QUANTITIES AS USED IN BABYLONIAN MATHEMATICS

Aldo Bonet (*)

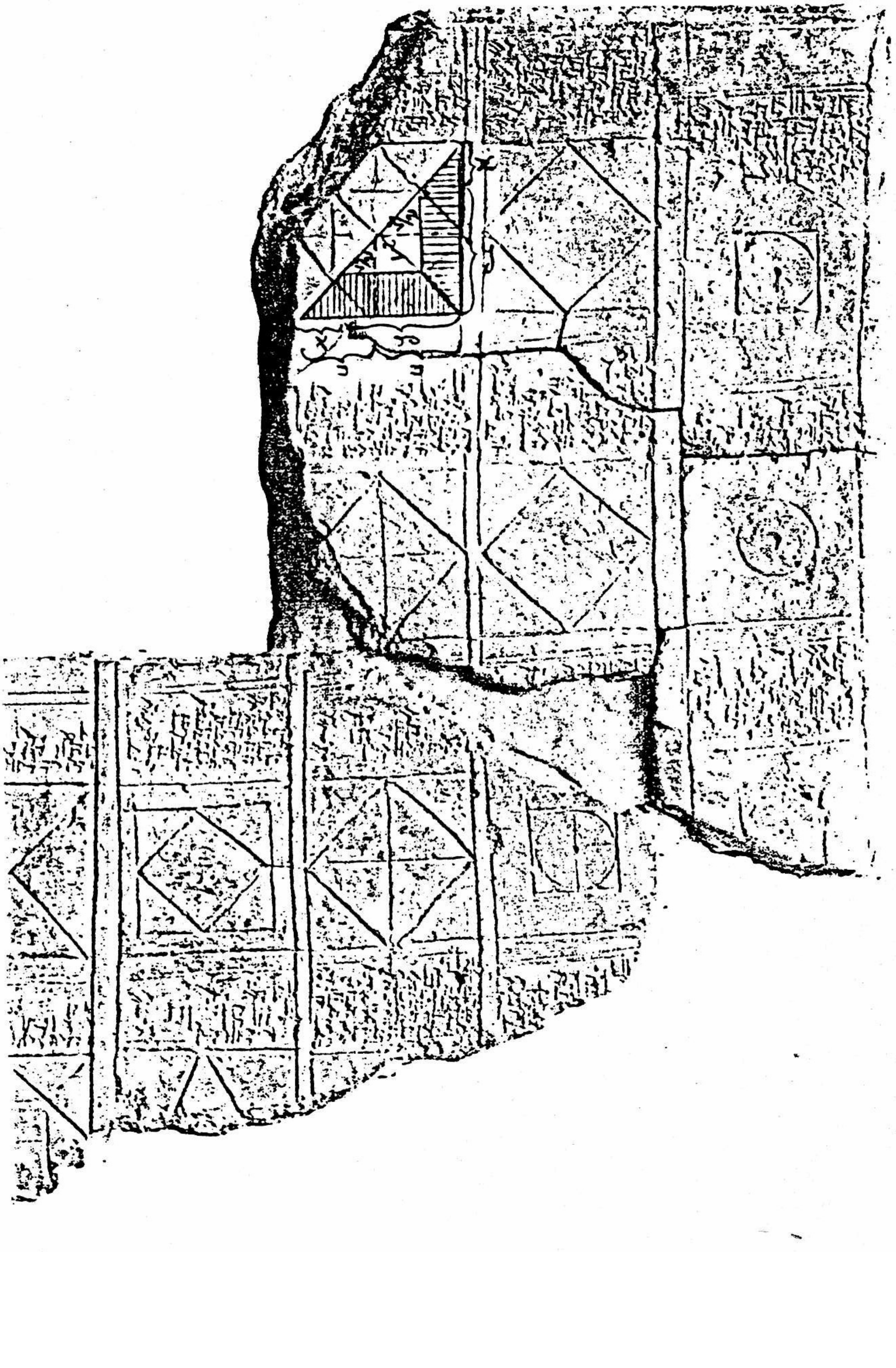
SUMMARY: A geometric study is proposed that forms the basis of the principle of half-sums and half-differences as used by the Babylonians. Diagrams pertaining to this study are developed, and applied in the demonstration of algebraic procedures of given problems from various recovered Babylonian tablets. The geometric study of half-sums and half-differences is further developed for third degree equations, extending to the geometric construction as hypothesized by S.J. Lurje (in the well-known study of quadratic sums). In conclusion, various didactic proposals on the subject are suggested.

Our knowledge regarding the Babylonian civilization comes from clay tablets marked with cuneiform characters.

The various recovered tablets[1] show that the Babylonians were able to resolve problems known to modern mathematics as the following types of equations: first degree equations, second degree equations, certain third degree equations, and higher degree equations that can be reduced to the second and third degree.

* The author's address: Street Guardini, 6 – 38100 Trento - Italy

[1] There are around 300 tablets containing mathematical content; some are from the Sumerian period (3000-2100 BC), others, forming a larger group, are from the period spanning the Hammurabi era until 1500 BC, while others date from the era of the new Babylonian empire and the Seleucid period (600-300 BC). The greatest contribution to the study of these tablets has been made by O. Neugebauer, F. Thureau-Dangin and E.M. Bruins.



Babylonian tablet from 1800 BC proposing various problems involving square roots

The solutions always refer to a particular case; there are neither formulas nor theorems given, although it seems that the Babylonians knew, without a doubt, general calculus rules.

On the BM 13901 tablet[2], which comes from the beginning of the second century BC, there are, as noted, twenty-four problems that can be classified in three groups.

The first two groups furnish a basic explanation which consists of two methods of solving the problems – that of completing the square in the case of a quadratic equation, and that of the half-sum and half-difference of unknown variables for a system. In effect, the first method could appear in the application of the second.

The half-sum and half-difference of roots play the role of the auxiliary unknown (or sub-root), and allow for simultaneously resolving both roots.

While it was widely applied by the Babylonians, the demonstration of the principle that regulates the half-sum and half-difference of roots has not been recovered in any other documents or tablets yet discovered. Consequently, neither has the algebraic identity widely used in the Babylonian mathematical writings.

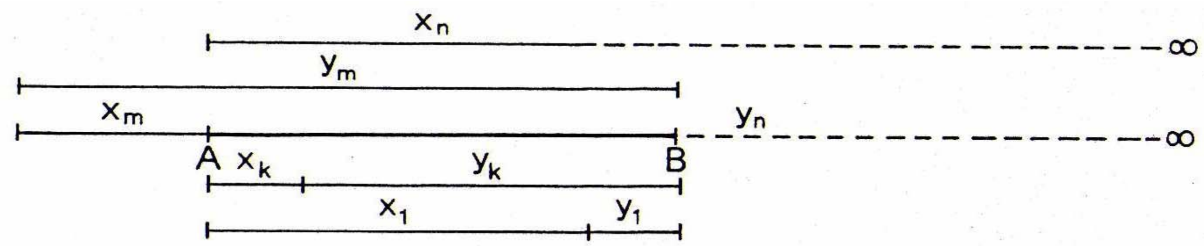
That which will now be demonstrated is the above-mentioned principle, which was probably fundamental to Babylonian theory in algebraic equations.

The theory could have taken its fundamental principle from the observation of a generic quantity represented by a segment of known width, and on which, to the end of determining the components of the unknown size with only the use of a ruler and a compass (both instruments already in use by the Babylonians), there could have been established the principle which regulates the half-sum and half-difference of unknowns, indispensable and above all of common character for the successive developments of equation systems.

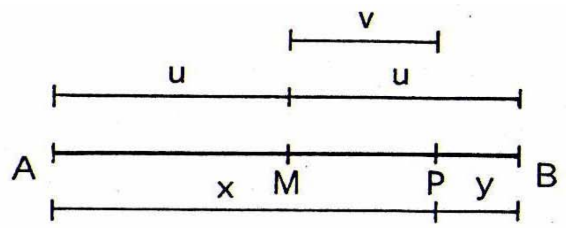
A segment of known width can always be interpreted as the result of a sum or difference of n unknown sizes.

[2] Caveing, 1982, I, page 91-95

To demonstrate the above, we proceed making reference to the case in which a segment is made up of only two unknown quantities (x, y); it can be observed that the segment, in this case, is the result of infinite combinations of two quantities, and therefore has infinite solutions.



Now we will examine how the values of a pair of quantities selected at random can be determined.



The quantities (or roots) in question are the unknowns (x, y) where «p» is a point on the segment.

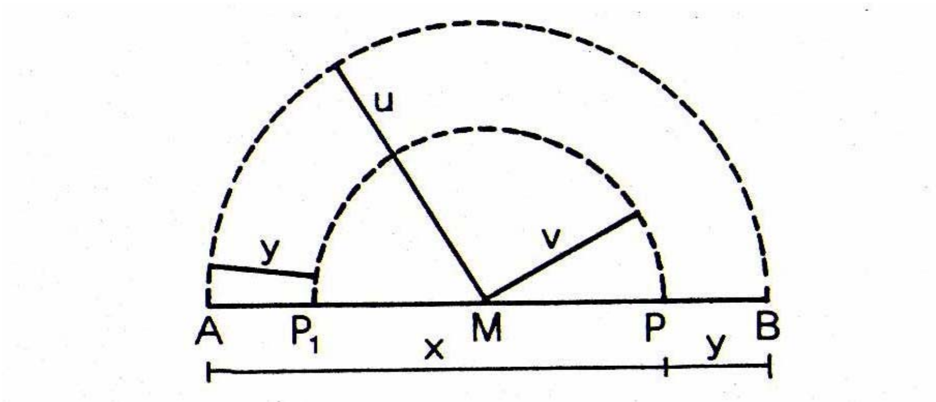
As can be observed, the quantities (x, y) can be broken in turn into two other fundamental lesser quantities (or sub-roots) which for our purposes will be named «u, v» and accordingly:

$$x = u + v;\ y = u - v$$

where «M» is the midpoint of the segment.

It can be noted that determining (x, y) is the equivalent of determining (u,v).

The sub-root «u» is already known since it is half of the segment $AB = x + y$, or to be precise, it is the half-sum of the unknowns (x,y), and the problem therefore remains to determine the remaining lesser quantity or sub-root «v». To this end it should be observed:



Transferring or rotating the unknown y by 180° on the segment AB with the compass point on the mid-point «M» there are two semicircles formed[3] with respective diameters equal to $2u = x + y$ and $2v = x - y$, or rather, «v» is the half-difference between segments AP and AP_1; therefore, $v = (x-y) : 2$

It can be deduced that:

A) the elementary resolution of the proposed problem can be obtained admitting notice of the sum of the two unknown quantities (x, y) as a necessary and sufficient condition, also their difference, or vice-versa.

A «system» can therefore be made, using algebraic concepts:

(1) $x + y = c; x - y = d$

which can be solved with the established principle of the half-sum and half-difference as follows:

$x, y = u \pm v; x, y = c/2 \pm d/2.$

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[3] From the book by Carl B. Boyer *History of Mathematics,* ISEDI, 1976, on p. 37: in a problem contained in a text from the ancient Babylonian period, we find two linear equations with two unknowns of the first degree, referred to in the text as “the first ring of silver” and the “second ring of silver” respectively. This way of expressing a pair or equations with words, “first and second ring” forms a system, and could probably have an illustrated geometric analogy where the rings are distinguishd with the respective radii: $u = (x + y) : 2$ and $v = (x - y) : 2$.

Analogous geometric representations are found in the Euclidic elements for the notations in the construction of a segment x for which $x^2 = ab$. Refer to p. 94 of Carl B. Boyer’s text.

To the same deduction (A), add that if the unknowns (x, y) were multiplied by a given number «a» or two different numbers (a, b) from the following respective systems:

$$ax + ay = c,\ ax - ay = d;\ ax + by = c,\ ax - by = d$$

which can be solved using the afore mentioned principles.

The Babylonians must have come to a probable hypothesis reached from these deductions. This is demonstrated by the fact that they frequently reduced a second-degree system (made of a sum and product) to a first-degree system (made of sum and difference).

For example, a quadratic equation such as $x^2 + b = ax$ with $b, a > 0$ and containing in the problem III from text IV from the Susa[4] tablets, which are derived from the system:

(2) $x + y -= a, \quad xy = b$

which is on the same tablet, and which they probably represented with the geometric diagram shown in figure 1.

Based on the previously made deduction (A), since the principle of half-sum and half-difference can be applied to determine the sub-roots, the Babylonian approach would have been based on determining the missing sub-root, in this case «v» since

$$u = (x + y) : 2 = a/2$$

is noted, and consequently leads the system of type (2) to a type (1) system.

To this end, transforming the previous diagram from figure 1 to the corresponding figure 2, it can be observed that the trapeziums are equivalent to the right-angled $xy = b$, and so:

$$(a/2)^2 = b + v^2;\ v^2 = (a/2)^2 - b$$

which leads to

$$v = \sqrt{(a/2)^2 - b} = (x - y)/2$$

[4] Bruins-Rutten 1961, p. 63-69

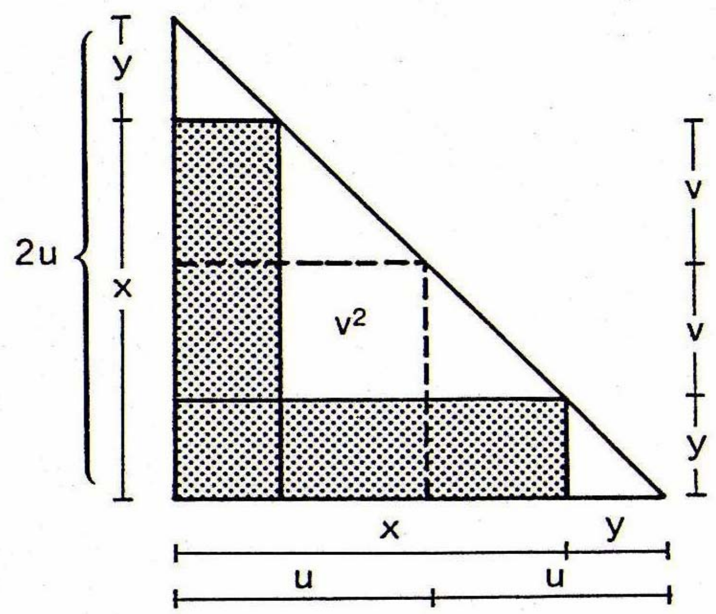

Figure 1.

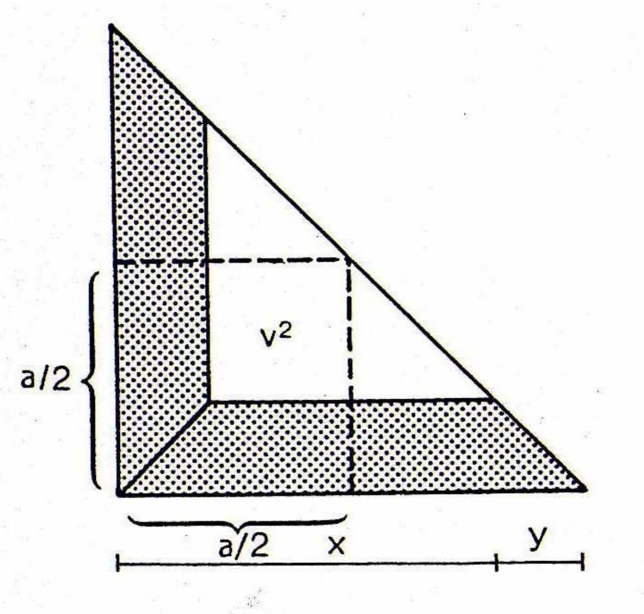

Figure 2.

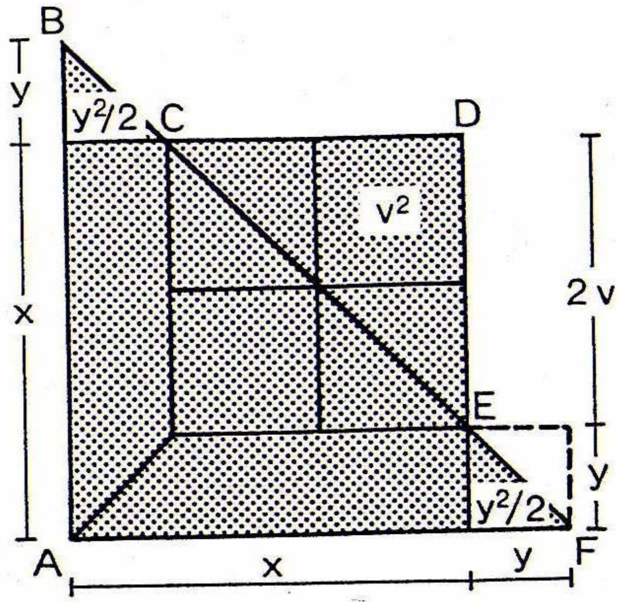

Figure 3.

knowing that $x,y = u \pm v$ results in:

$$x = a/2 + \sqrt{(a/2)^2 - b}; y = a/2 - \sqrt{(a/2)^2 - b}$$

which are the effective algebraic steps taken by the scribe.

The Babylonians never took into account negative solutions, perhaps because the problems remained in the realm of geometric quantities (always positive) as can be recognized in the diagrams previously displayed.

With the same diagrams, it is possible to translate algebraically the steps that lead to the solution of problem 1 from the prism AO 8862[5] in which the scribe made use of the identity:

$$(x + y)^2 : 2 - (x - y)^2 : 2 = 2\,xy$$

recognizable in the diagrams themselves.

The same can be said for problems 1 and 2 from the BM 13901 tablet; although the steps recorded by the scribe may seem like the application of completing the square, it shouldn't be excluded that the problems were solved with the half-sum and half-difference principle. In this way, they would have been tackled from a systematic point of view in order to derive the equation proposed, then proceeding to algebraic steps through the visualization of the previously examined diagrams such that, with notable practicality, the solution of the equations would be reached.[6]

The algebraic steps suggested by the diagrams recognize that, when the sub-root «*u*» is extracted in the problem, that is, in the event that the sum of the unknowns is known, the expression under the root in the procedure of solving «*v*» should appear as a subtraction; the minuend, in the opposite case: «given *v* solve *u*»; that is effectively what was recorded by the scribe.

In following with the diagrams thus seen, it was possible to establish the known identity

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[5] Prism AO 8862, currently housed in the Louvre, dates from the Hammurabi era (circa 1700 BC), and contains 8 problems. It was studied by O. Neugebauer (1937) and by F. Thureau-Dangin (1938).

[6] This is not to exclude the supposition by Van Der Waerden for the notable products, since the diagrams are in perfect harmony with that maintained by the same Van Der Waerden; however, for problems 1 and 2 on tablet BM 13901 it was desireable to follow the path for which it doesn't matter if the Babylonians knew about the square of binomials or not.

as known by the Babylonians:

$$(x + y)^2 : 2 + (x - y)^2 : 2 = x^2 + y^2 ,$$

and therefore from the examination of the geometric translation of the system:

$$x + y = a, \qquad x^2 + y^2 = b$$

and from the diagram shown in figure 3 which establishes:

$$A\,B\,F = (x + y)^2 / 2;\ C\,D\,E = (x - y)^2 / 2;$$

$$A\,B\,C\,D\,E\,F = x^2 + y^2,$$

to be precise:

$$A\,B\,C\,D\,E\,F = A\,B\,F + C\,D\,E,$$

that is:

$$x^2 + y^2 = \frac{(x+y)^2}{2} + \frac{(x-y)^2}{2}$$

With analogous reasoning and diagrams, it was possible for the Babylonians to translate algebraically the identity they used in problem 9 on the BM 13901 tablet, that is to say, the following:

$$\frac{x^2 + y^2}{2} = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2$$ [7]

Following the same paths of reasoning so far developed, it certainly wasn't difficult for the Babylonians to move from a second degree problem to one in the third degree; it was sufficient to find the solution of the sides of a parallelepiped with a squared base, noting the sum of the length and the height and volume, a problem which is equivalent to the following system:

$$x^2 y = a,\ x + y = b;$$

We represent the problem geometrically in this case with the perspective,

[7] Giacardi L. 1985, *Alle origini dell'algebra*. From the "recipes for calculus" (Egyptians, Babylonians) to the rigours of geometric algebra (Greeks). Actions from the mathematics meetings, Proveditorato agli Studi di Grosseto, p. 21-46

but for the Babylonians it was more practical to mix the clay directly to obtain forms such as parallelepipeds, cubes, etc. or to form, for example, quadratic tables with a side where $x + y = b$, that is to say:

$$(x + y)^2 = b^2;$$

six of these forms joined together would form a cubic box with a side $x + y = b$, in which the total volume would be $x^2y = a$.

This line of reasoning can be followed to demonstrate how one arrives at the process of construction in space as hypothesized by S. J. Lurje[8] in his noted demonstration on the sum of squares; the fact of adding the same construction with completely different subjects certainly reinforces the hypothesis of S.J Lurje.

Perspectively, the given problem is presented as in figure 4.

It can be seen and experimented that the missing sub-root «*v*» cannot be determined with the equicomposition of a prism through the known elements, and therefore the obstacle posed by problem can only be “overcome” with the assistance of an empirical table which lists the values to attribute to the unknowns corresponding to the known volumes.[9]

Certainly, as industrious as the Babylonians are known to have been, they would have noticed a simple addition which could have been applied to the above problem, extracting in this way the value of «*v*» and therefore posing and summing along with the volume $x^2y = a_1$ the corresponding volume: $y^2x = a$, and in this way arriving at the following system:

(3) $x^2y + y^2x = a, x + y = b$

[8] S.J. Lurje, *Archiimedes*, Vienna 19948, p. 17

[9] The Babylonians solved equations in the form $x^3 + x^2 = a$ with the use of tables, one of which was discovered and, as noted, expresses the values of $n^3 + n^2$ with n variable from 1 to 30.

Notice that for the Babyloinians, the use of a makeshift empirical in order to solve the above cubic equations would have been a rather undesireable solution, as it would have disturbed the harmony of their logic process characteristic of algebraic equation theory. It should be pointed out that to try, nonetheless, to find an elegant algebraic solution for such a daring problem would have been quite a puzzle for the Babylonians, and perhaps a challenge that would have spanned generations. Only in 1500 AD was such a problem proposed and solved by Italian algebrists – an important moment in the history of mathematics.

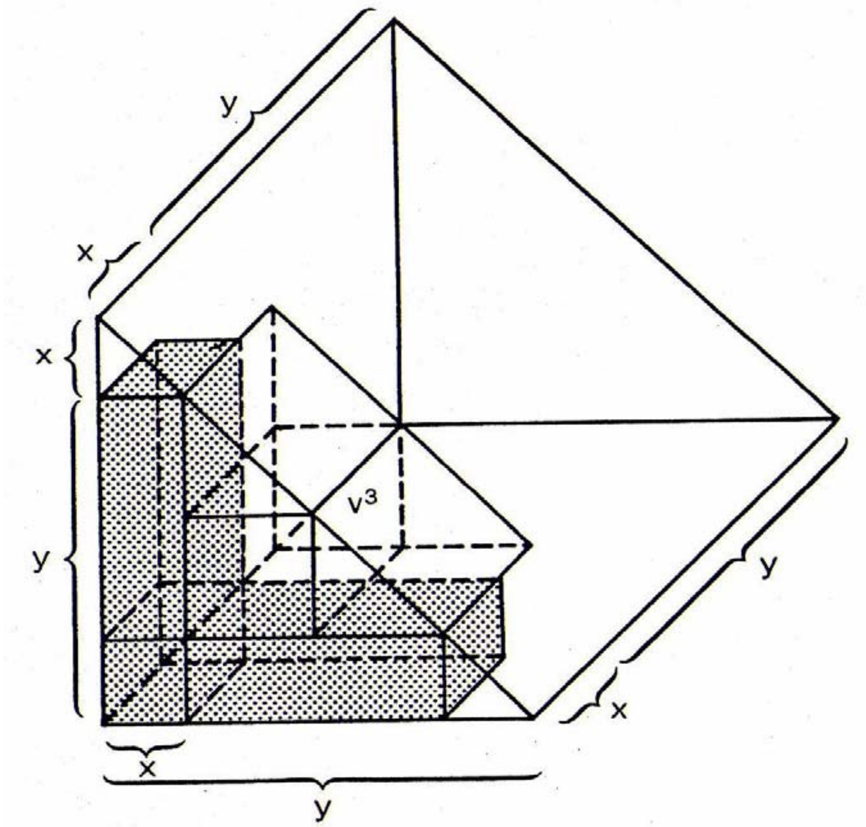

Figure 4.

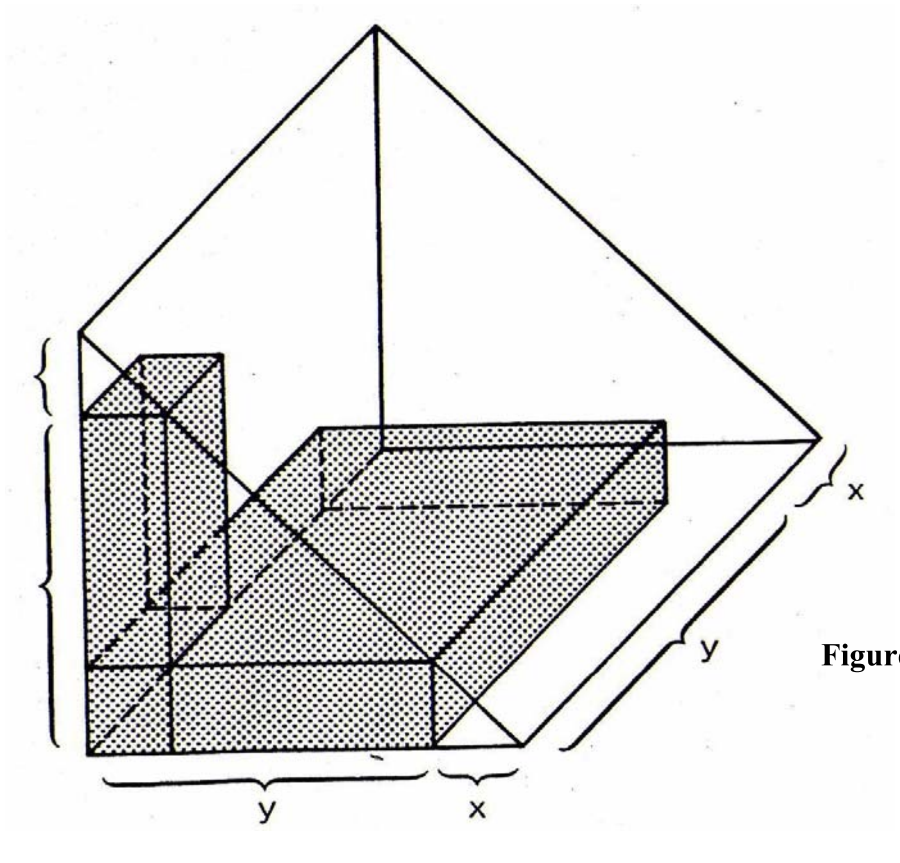

Figure 5.

which could be transformed in the noted symmetric second degree system which they knew how to resolve.

The perspective representation of system (3) is represented in figure 5. The prism can be taken apart as the sum of two parallelepipeds with the dimensions: $xy\ (x+y)$ plus a prism with a section 2 v^2 and length equal to: $(x + y) = b$, creating a composite diagram given in figure 6.

The volumes of the two trapezoid section solids are equivalent in volume to the two parallelepipeds, each of which is the sum of:

$$x^2y + y^2x = a$$

In this way, system (3) is solvable since the missing sub-root «v» can be determined. The algebraic steps to reach the desired solution are suggested and thus recognized in the previously cited diagrams.

Further, the Babylonians solved problems with multiple unknowns, and it should not be excluded that they considered more general or more complex problems as an exercise, leading to the following system:

$$x^2y + y^2x = a; x_1^2y_1 + y_1^2x_1 = c; x_2^2y_2 + y_2^2x_2 = b^3/4;$$

(4) $$x + y = x_1 + y_1 = x^2 + y^2 = b$$

in other words, the sums of parallelepipeds with different volumes but with equal sums of the respective sides of the bases and the respective heights are known, a sum which is equal to: «b» (and thus isoperimeters) is used to calculate the bases and heights of the respective parallelepipeds.

A solvable system, since it is sufficient to apply the resolution procedure used for system (3) and repeat it for each of the equations in system (4).

The perspective representation for system (4) is given in figure 7, and it is analogous to the construction proposed by S.J. Lurje in the demonstration of the formula which allowed the Babylonians to determine the sum of squares of consecutive whole numbers.[10]

The above-given procedure and that most likely used by the Babylonians to solve the problems on the clay tablets leads us to observe that their development was different from that of the Arabs and Greeks,

[10] Giacardi L., Roero S.C. 1979, *La matematica delle civiltà arcaiche, Egitto, Mesopotamia, Grecia*, Stampatori, Turin, p. 132 onward

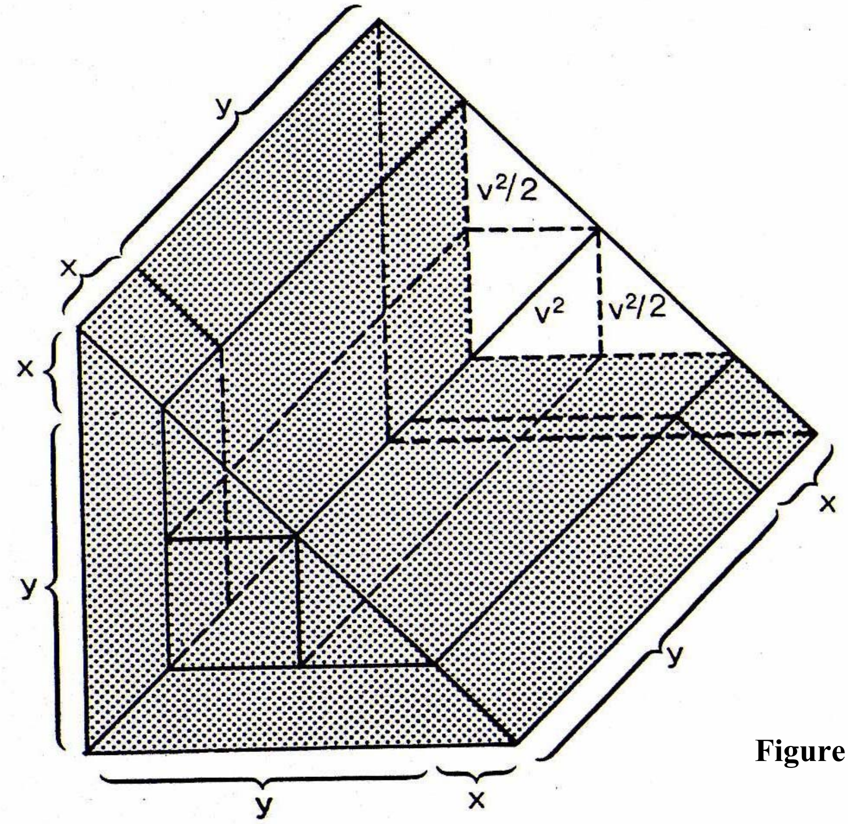

Figure 6.

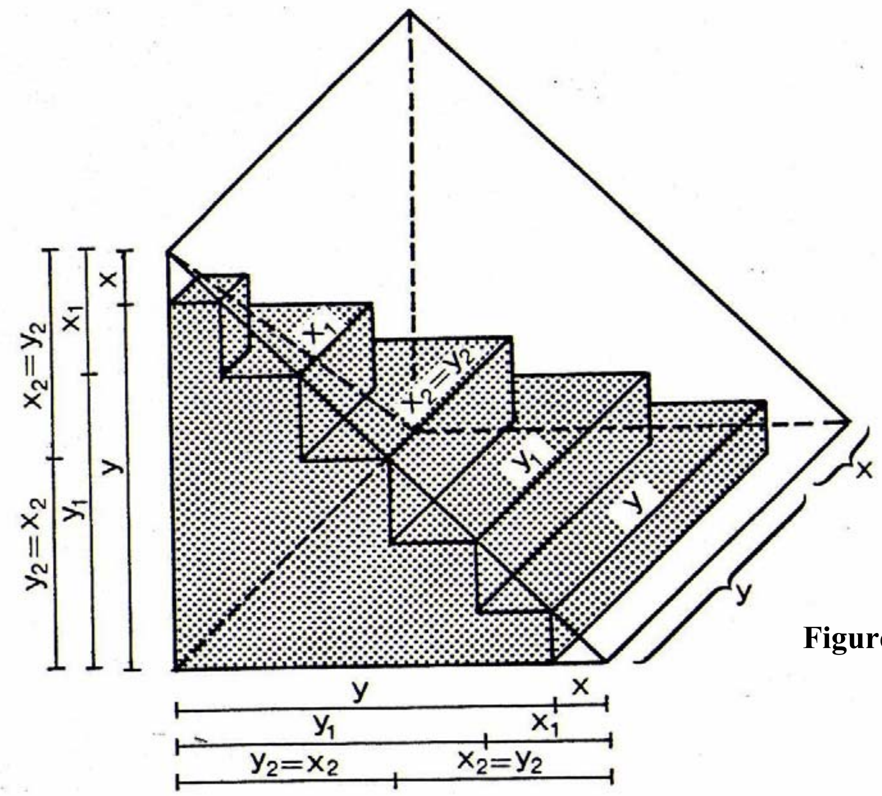

Figure 7.

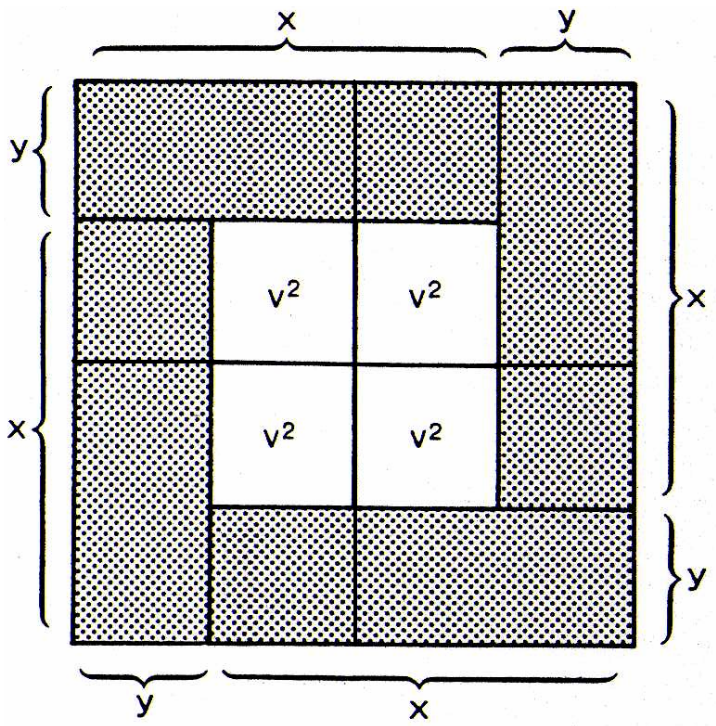

Figure 8.

in that the basic concept was uniquely based on the research (using preliminary diagrams) of the missing sub-root «*v*» as required.

Finally, through diagrams similar to those already seen, but which don't make provisions for surfaces and volumes, it can be shown that the Babylonians could reach analogous conclusions to those elaborated for previous problems, to be precise putting forward once again problem III from text IX from the Susa tablets which we know contained the following system:

$x + y = a, xy = b;$

In this case, determining «*v*» is recognizable from the diagram given in figure 8, which shows the algebraic steps which must be undertaken in order to determine the sub-root «*v*», and thus the solution of the system.

From the same diagram, it is possible to foresee and establish the following identity as known and applied by the Babylonians:

$(x + y)^2 = 4\,xy + (x - y)^2$

It follows that:

$$(x - y)^2 = (x + y)^2 - 4xy$$ [11]

Bearing in mind the schematic diagram in figure 8, a third degree problem which leads to the following previously seen system, that is:

$$x^2y = a,\ x + y = b,$$

would consequently be represented geometrically by the diagram given in figure 9. The two volumes, one above and one below, interlock perfectly in such a way as to form a cube with sides: $x + y$.

At first examination, the solution to determine the sub-root «v» would seem quite easy, but unfortunately we know that it isn't really so, since the lower volume in the diagram is an unknown. For this reason, at least from an algebraic point of view, the diagram remains unsolvable.

It is known that the Babylonians constructed miniature models in clay when planning the construction of buildings, but it remains unclear whether they also may have made models in order to solve more difficult problems (such as that cited above), which in turn could have influenced the Babylonian architecture... the Tower of Babel could perhaps be considered an example.

The whole subject that we have examined can be considered as a valid teaching aid for educational purposes.

The fundamental concept of the Babylonian theory for algebraic equations is, as we have seen, the principle that regulates the half-sums and half-differences of unknowns.

The initial geometric impetus offers the advantage of being able to understand in a visible and immediate measure the key introduction for the successive study of equations.

The theory as used by the Babylonians allows us to see and make use of for didactic purposes the simplicity and uniformity of the procedure with basic logic, from first-degree through higher-degree equations; This is a procedure that is presented above all

[11] Giacardi L., Roero S.C. 1979, *La matematica delle civiltà arcaiche, Egitto, Mesopotamia, Grecia*, Stampatori Turin, p. 132 onward

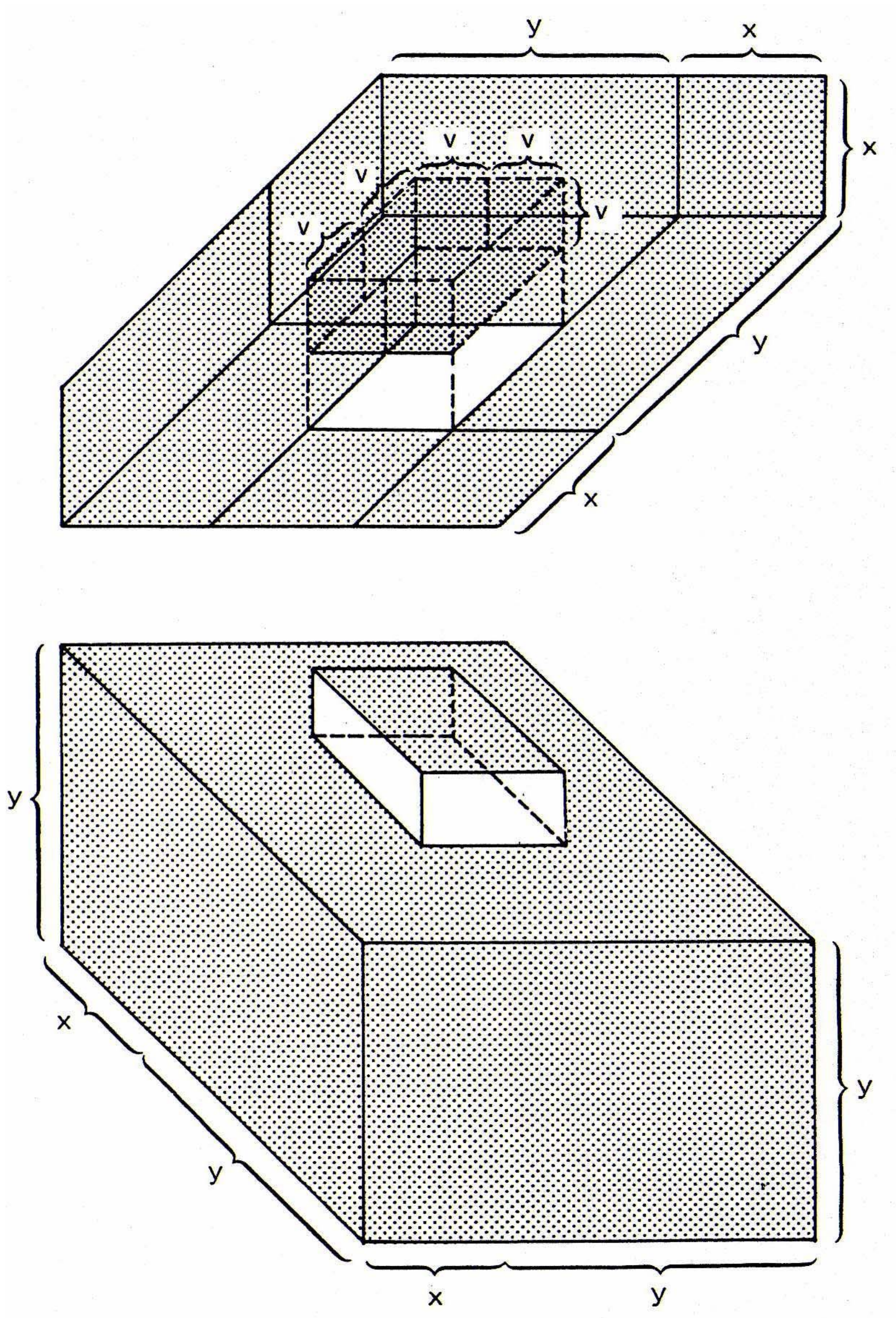

Figure 9.

by a common system for first, second and third degree equations, something that is treated in an altogether different manner in text books commonly used in Italian two-year post-secondary school courses.[12] This gives us the net sensation that equations of different degrees are categories of independent equations without anything in common, the exception being that they are both equations and they are tackled with different specific methods.

The procedure used by the Babylonians suggests to us that the study of equations can be seen as in different ways, and it also makes us reason that in the end, solvable equations, while they may be of different degrees, all have in common as a species, if they can be considered such, a common genealogy. As such, if the equations are solvable, it is because they aren't anything more than equivalent forms that are more highly evolved from their basic and simpler systems, and of lower-degree than those from which they have descended.[13]

The systems established and solved in the first pages of this treatise are as follows:

(1) $x + y = c, x - y = d;$

(2) $x + y = c/a, x - y = d/a$

(3) $ax + by = c, ax - by = d$

They are all fundamental systems, basic for other solvable systems, also of higher than the first-degree; if they are solvable, it is because they can be traced back to, and therefore considered equivalent to, the fundamental established systems.

From the Babylonian theory, based on the deduction (A) it can be defined that an elementary first-degree system of two unknowns is compatible if the equations making up the system are conditionally and necessarily expressed one by a sum and the other by a difference of two monomers of the unknowns x and y.

As we have seen, the solution is determined by resolving the sub-roots

[12] Above all considering the textbooks used by technical institutes such as *Aritmetica a Algebra* and *Algebra e Geometra Analitica* by Prof. Giuseppe Zwirner for the two-year university course in the Istituti Tecnici er Geometri. Edizioni Cedam. Padova

[13] This is an interesting didactic point, suggested from the Babylonian theory of algebraic equations, and that we recommend to teachers since students, in general, are attracted by and find satisfying a simple and exhaustive discussion that leads to rapid solution of equations starting from a uniform principle such as that of the Babylonians. It also explains why some equations are solveable while others belong to the unsolveable category.

(u,v), that can be obtained through substitution.

Sub-roots of unknowns can also be substituted in systems that are higher than the first-degree.

Before going to higher degree systems, we first consider a known first-degree system in which the coefficients are not proportional in terms of absolute values, that is, the following:

(4) $ax + by = c,\ a'x + b'y = c'$

In this problem, either of the two equations from system (4) can be considered. Taking the first system, the following is proposed:

$$x = (u + v)/a; \qquad y = (u - v)/b$$

By substitution, one obtains:

$$u = c/2,\ u\,(a'/a + b'/b) + v\,(a'/a - b'/b) = c'$$

and therefore:

$$x = (bc' - b'c) : (a'b - ab');\ y = (a'c - ac') : (a'b - ab')$$

These are the effective substitutions for system (4).

If the system is solvable, that means that it is equivalent to one of the fundamental systems, and, in effect, system (4) is equivalent to the basic system (3). If, in the following expression:

(5) $u\,(a'/a + b'/b) + v\,(a'/a - b'/b) = a'x + b'y = c''$,

the first member of (5) is reduced to $2v$, the second leads inevitably to a first-degree equation of two unknowns expressed by the difference of two monomers having as their coefficients the same values as the first equation in system (4), and the third member to an expression of known terms which are considered equal to «d». In conclusion, system (4) leads to system (3), a fundamental and basic system.

Proceeding by order of degree, we can now apply the aforesaid to a known second-degree symmetric system from which can be derived the known complete second-degree equations:

$$ax^2 + bx + c = 0;\ ay^2 + by + c = 0$$

and therefore the system:

$xy = c/a, \qquad x + y = -b/a,$

for which is placed:

$x = u + v; \qquad y = u - v$

one obtains:

$u^2 - v^2 = c/a, \qquad u = -b/2a,$

and therefore:

$$x_{1,2} = \frac{-b + \pm\sqrt{b^2 - 4ac}}{2a} \qquad y_{1,2} = \frac{-b - \pm\sqrt{b^2 - 4ac}}{2a}$$

In other words:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad y_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which are the effective and known formulas used to solve for second-degree equations.

One can spontaneously make a connection between the roots and coefficients of the equations, and further, it can be verified that the symmetric second-degree system is equivalent to the basic system (2).

Like the conclusion to quadratic equations, we want to show how the above resolution formulas are a particular form of a more general formula; to further develop the above:

$u^2 - v^2 = xy = c/a, \quad x + y –b/a,$

we multiply the first equation by four, obtaining:

$4\,u^2 - 4\,v^2 = 4\,xy = 4\,c/a, \quad x + y = -b/a;$

In other words:

$$(x+y)^2 - (x-y)^2 = 4xy = 4c/a, \qquad x + y = -b/a$$

Putting in place the exponents from the first equation that are equal to «n», we have:

$$(x+y)^n - (x-y)^n = 2\left[\binom{n}{1}x^{n-1}y + \binom{n}{3}x^{n-3}y^3 + \binom{n}{5}x^{n-5}y^5 + \ldots\binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n\right] = 2^n c/a,$$

$$x + y = -b/a$$

If «n» is an even number, remove the term

$$\binom{n}{n} y^n,$$

If, however, it is an odd number, remove:

$$\binom{n}{n-1} x y^{n-1};$$

System (6) is solved with the following generalized formula:

$$x = \frac{-b + \pm\sqrt[n]{(-b)^n - 2^n a^{n-1} c}}{2a}; \qquad x = \frac{-b - \pm\sqrt[n]{(-b)^n - 2^n a^{n-1} c}}{2a};$$

When the index of the root is odd, remove the ± sign and highlight that of the root; for $n = 2$, the resolving formula is obtained from the completed second-degree equations already examined.

Finally, we would like to examine a third-degree equation, from which the following known equation is derived:

$$x^3 + qx + r = 0; \qquad q^3 y^3 - 3rq^2 y^2 + (3r^2 q + q^4)\, y - r^3 = 0$$

That is, the following system:

$$x^3 + qy = 0, \quad x - y = - r/q;$$

substituting the sub-roots for the unknowns, we obtain:

$$u^3 + v^3 + 3\, uv\, (u + v) + q(u - v) = 0, \qquad 2v = - r/q$$

It can be observed that the first equation is completed and of the third-degree in «u», therefore, a solution is not possible since it can not be reduced to one of the fundamental systems.

The only way to obtain the first equation with only the sub-root of «u» is by considering a third equation or condition to the system, that is:

$$uv = -q/3,$$

and, in fact, the new system is obtainable with three equations:

$u^3 + v^3 + r = 0, \qquad uv =- q/3, \qquad v =- r/2q.$

Now we have two expressions for «*v*», so we must chose that which satisfies the given conditions, namely:

$uv = -q/3;$ that is: $v =- q/3u,$

from which one obtains

$$u = \sqrt[3]{-r/2 \pm \sqrt{r^2/4 + q^3/27}}$$

Keeping in mind the following:

$v3 =- r - u^3$

one can obtain:

$$v = \sqrt[3]{-r/2 \mp \sqrt{r^2/4 + q^3/27}}$$

For distinction, we choose the superior algebraic sign, leading us to:

$$x = \sqrt[3]{-r/2 + \sqrt{r^2/4 + q^3/27}} + \sqrt[3]{-r/2 - \sqrt{r^2/4 + q^3/27}}$$

$$y = \sqrt[3]{-r/2 + \sqrt{r^2/4 + q^3/27}} + \sqrt[3]{-r/2 - \sqrt{r^2/4 + q^3/27}} + r/q$$

The solving formulas are those taken from fifteenth century mathematicians Tartaglia e Cardano, but it is surprising that the principle of the half-sum and half-difference of unknowns is, once again, the most important feature of the problem, which leads to the solution of the third-degree equations which have so absorbed mathematicians from the Renaissance[14].

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[14] The fact that the same principle can be used to approach even completely reduced third-degree equations in the known canonic form with the same logic and difficulty as the preceeding problems leads us to believe that the introduction of same as a subject of study for middle and upper school can, in fact, be accessible. Further, the same Babylonian principle of the half-sum and half-difference of unknowns can be used to solve certain higher-degree problems typically encountered in texts currently in use. For example: $x^3 + y^3 = b, x + y = a;$ $x^2 + y^2 = a, xy = b.$ Keep in mind that, as a curiosity, the solving formula for y presents an analagous form, today known as a "solving formula of the third-degree equation resolution" of the fourth-degree equation. This was discoverered by Ludovico Ferrari at the request of Gerolamo Cardano, and it was published in his treatise, *Ars Magna*.

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